Selected Classical Problems in the History of Mathematics

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Contents

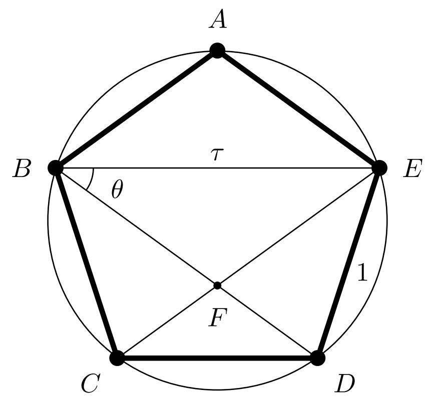

Figure 1: A regular pentagon

1 The Regular Pentagon

a.) Show △EDF ≃△BDF in Figure 1.

Begin by observing that the marked angle θ is equivalent to ∠FEB, ∠DEF, ∠EBD, ∠EDA, and ∠BDA because each angle covers a segment with an equivalent central angle. Also observe that ∠EFB is 2θ less than two right angles (because the interior angles of a triangle sum to two right angles). Next, observe that ∠EFD is 2θ because it is adjacent to ∠EFB. Surely ∠EDF and ∠DEB are also 2θ because they are sums of two angles of θ. Taken together, we can conclude that △EDF ≃△BDF because of angle-angle-angle similarity. □

b.) Conclude $\frac{\tau-1}{1} = \frac{1}{\tau}$ and $\tau = \frac{1+\sqrt{5}}{2}$.

Let us give the name τ the entire diagonal $\overline{BD}$. From our previous result we know △EDF ≃△BDF. Similar logic can show that △BCF is similar to these as well. Note, that with $\overline{ED}$=$\overline{BC}$ we can conclude △EDF ≅ △BFC, implying that the length of $\overline{BF}$ is 1. Furthermore, we know the length of $\overline{ED}$ is to $\overline{DF}$ in the same ratio as $\overline{BD}$ is to $\overline{DE}$ by triangle similarity. In other words, 1 is to $\tau - 1$ in the same ratio as τ is to 1. □

If we rewrite the relation above after cross multiplying terms we find $\tau(\tau - 1) = 1$ or $\tau^2 - \tau - 1 = 0$. By the well known quadratic formula, this relation is satisfied by $\tau = \frac{1+\sqrt{5}}{2}$. □

c.) Construct τ and given a unit segment construct a regular pentagon.

The construction below is depicted in Figure 2.

1. Label the points on the unit segment O and A (O as the origin).

2. Extend the line out from O past A.

3. Copy the unit segment five times past A on this line and mark point B.

4. Form C from the midpoint of $\overline{OB}$.

5. Construct a circle with C is its center and $\overline{CB}$ as its radius.

6. Construct a line perpendicular to $\overline{OA}$ at A.

7. Mark the intersection of the perpendicular line and the circle D. *The segment* $\overline{AD}$ *is the geometric mean of* $\overline{OA}$ *and* $\overline{AB}$*, or exactly* $\sqrt{5}$.

8. Copy the length of $\overline{AD}$ along the first line starting from A and call its terminal point E.

9. Form F from the midpoint of $\overline{OE}$. *The segment* $\overline{OF}$ *has length* $\frac{1+\sqrt{5}}{2}$*, exactly* τ.

10. Construct circles from A and O (*one unit part*) with the length $\overline{OF}$ (τ).

11. Call the intersection of these circles G. *Note, G is the top of the pentagon.*

12. Construct a unit circle centered at O and call its intersection with the unit circle centered at A closest to G the point H.

13. Construct a unit circle centered at A and call its intersection with the unit circle centered at O closest to G the point I.

14. Connect O, A, I, G, and H with line segments. *These line segments form a regular pentagon with unit side-length.*

d.) Show $\tau = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\dots}}}$**.**

We have shown that $\frac{\tau-1}{1} = \frac{1}{\tau}$. Observe that this is simply $\tau - 1 = \frac{1}{\tau}$. Isolating τ by adding 1, we have have the following recursive definition:

$$\tau = 1 + \frac{1}{\tau}$$

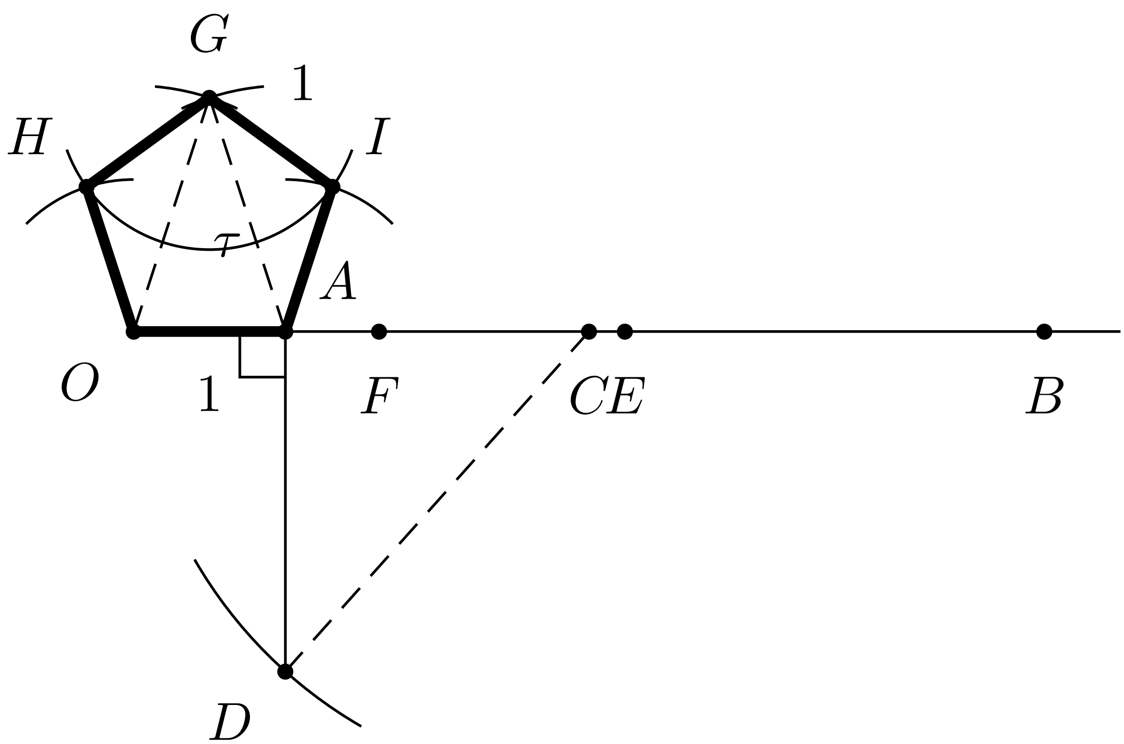

Figure 2: Construction of a regular pentagon

Repeated substitution confirms the desired result.

$$\tau = 1 + \frac{1}{\tau}$$

$$\tau = 1 + \frac{1}{1 + \frac{1}{\tau}}$$

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$$\tau = 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{\tau}}}}}}$$

$$\vdots$$

$$\tau = 1 + \frac{1}{1+\frac{1}{\tau}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}$$

and so on.

$\square$

Also of interest, we can rewrite the above relation to give this alternative recursive definition:

$$\tau = \frac{1}{\tau - 1}$$

Yielding:

$$\tau = \frac{1}{\frac{1}{\frac{1}{\cdots - 1} - 1} - 1}$$

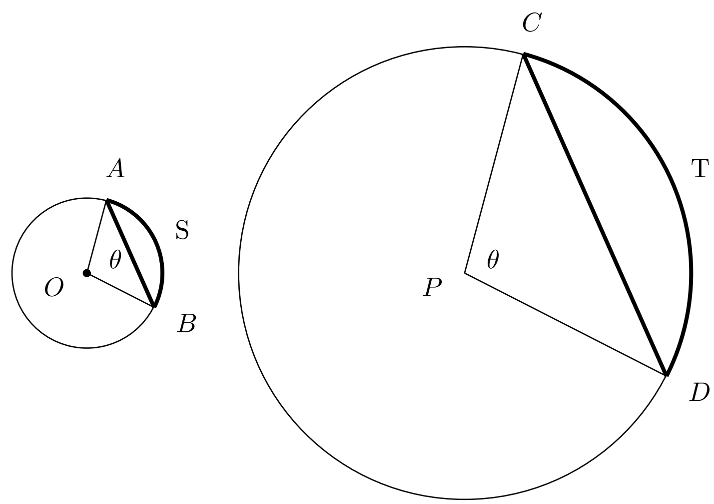

Figure 3: Similar segments

2 Circular Segments and Lunes

a.) Show that similar segments are to one another as squares on their chords.

Accept without proof the easily verifiable claims that (1) *similar sectors are to one another as squares on their radii* and that (2) *like sides sides of similar triangles are to one another as other like sides on those triangles.* Now, observe that △OAB is similar to △PCD because both are isosceles (two legs are radii) and they have a common angle θ. By (2) we know that $\overline{OA}$ is to $\overline{PC}$ as $\overline{AB}$ is to $\overline{CD}$. From (1) we know segment S is to segment T as $\overline{OA}$ is to $\overline{PC}$. But this is precisely the ratio of $\overline{AB}$ is to $\overline{CD}$. Taken together, we can see that segment S is to segment T as $\overline{AB}$ is to $\overline{CD}$, or that *similar segments are to one another as squares on their chords.* □

b.) Show that the area of lune ⌢ ABC is the same as the area of △ABC, and hence is squarable. (Figure 4)

Consider the area of ⌢ ABC, surely it is exactly the area of △ABC *plus* the area of segments P and Q *minus* the area of segment R. Observe that segments P, Q and R are similar because they all share a right triangle at the center of their circles. Furthermore, since P and Q are similar and have identical chords, they are congruent. By our previous result, we know that segment R is to segment Q as a square on $\overline{AC}$ is to a square on $\overline{AB}$. The square of $\overline{AC}$ is 2 and the square of $\overline{AB}$ is 1. This implies that the area of segment R is twice that of segment P, or simply the sum of P and Q. Taken together with our expression for the area of the lune, we see that the areas added and subtracted from the triangle to form the lune annihilate, leaving only the area of the original triangle. Thus, the area of the lune and triangle are equal. Furthermore, since the triangles are squarable the lune is squarable as well. □

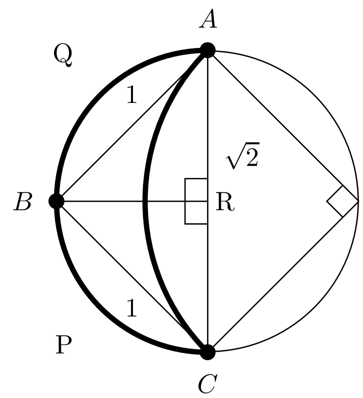

Figure 4: The lune ⌒ ABC

c.) Show that ⌒ ABCD is squarable. (Figure 5)

Consider the area of the ⌒ ABCD, surely it is exactly the area of the trapezium ABCD *plus* the area of segments T, U, and V *minus* the area of segment W. Following similar reasoning as in our last proposition, observe that segments T, U, V and W are similar because they share an angle of θ at the center of their circles. Furthermore, observe that since the smaller segments T, U, and V are similar and have identical chords, they are congruent. By our first result, we know that segment W is to segment T as a square on $\overline{AD}$ is to a square on $\overline{AB}$. The square of $\overline{AD}$ is 3 and the square of $\overline{AB}$ is 1. This implies that the area of segment W is thrice that of segment T, or simply the sum of the areas of T, U, and V. Taken together with our expression for the area of the lune, we see that the areas added and subtracted from the trapezium to form the lune annihilate, leaving only the area of the original trapezium. Thus, the area of the lune and trapezium are equal. Furthermore, since the trapezium can be decomposed into two squarable triangles the lune is squarable as well. □

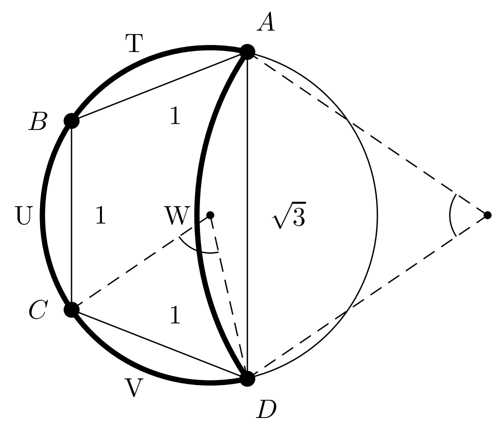

Figure 5: The lune ⌒ ABCD

3 Sum of a Geometric Series

Given a series of areas A,B,C,...,Z of which A is the largest and each is equal to four times the next in order. Show:

$$\mathbf{A}+\mathbf{B}+\mathbf{C}+\cdots+\mathbf{Z}+\frac{\mathbf{1}}{\mathbf{3}}\mathbf{Z}=\frac{\mathbf{4}}{\mathbf{3}}\mathbf{A}$$

Define $b=\frac{1}{3}B$, $c=\frac{1}{3}A$, ..., $z=\frac{1}{3}Z$.

$$\begin{aligned}
A+B+C+\cdots+Z+\frac{1}{3}Z &= A+B+C+\cdots+Z+\frac{1}{3}Z+(b+c+\cdots+z)-(b+c+\cdots+z)\\
&= A+(B+b+C+c+\cdots+Z+z)+\frac{1}{3}Z-(b+c+\cdots+z)\\
&= A+(\frac{4}{3}B+\frac{4}{3}C+\cdots+\frac{4}{3}Z)+\frac{1}{3}Z-(b+c+\cdots+z)\\
&= A+(\frac{1}{3}A+\frac{1}{3}B+\cdots+\frac{1}{3}Y)+\frac{1}{3}Z-(b+c+\cdots+z)\\
&= \frac{4}{3}A+(b+\cdots+\frac{1}{3}y)+\frac{1}{3}Z-(b+c+\cdots+z)\\
&= \frac{4}{3}A+\frac{1}{3}Z-z\\
&= \boxed{\frac{4}{3}A}
\end{aligned}$$

□

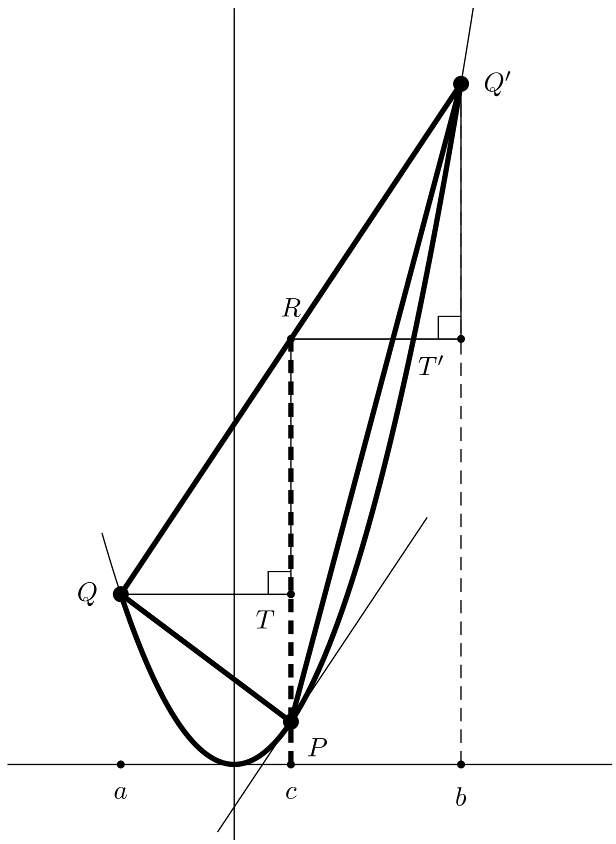

Figure 6: An arbitrary parabolic segment

4 Parabolic Segments

The problems below refer to Figure 6.

a.) Show using calculus that if $\overline{QQ'}$ is parallel to the tangent line at c then a vertical line through P bisects the chord $\overline{QQ'}$.

Let the coordinates of Q be (a, a^2) and the coordinates of Q' be (b, b^2). Any line parallel line to $\overline{QQ'}$ must have the same slope as $\overline{QQ'}$. Let this slope be $m = \frac{b^2-a^2}{b-a} = \frac{(b-a)(b+a)}{b-a} = b + a$. The slope of the tangent line at a point x on the parabola is $2x$. The point c where the tangent to the parabola is parallel to $\overline{QQ'}$ is simply $c = \frac{m}{2}$. In more detail, we see $c = \frac{b+a}{2}$ is the midpoint of these values. The right triangles △QRT and △RQ'T' are congruent (SAS) so $\overline{QR} = \overline{RQ'}$. Thus, the line through P bisects the chord $\overline{QQ'}$. □

b.) Show using calculus that area of the parabolic segment is $\frac{4}{3}\triangle PQQ'$.

Observe that the coordinates of R are $(\frac{b+a}{2}, \frac{b^2+a^2}{2})$ and the coordinates of P are (c, c^2).

Let $h(x) = (a+b)x + (a+b)a + a^2 - x^2$ be height of a vertical line intersecting the parabolic segment at any position x.

$$
\begin{aligned}
A_{\triangle PQQ'} &= A_{\triangle QRP} + A_{\triangle Q'RP} && \text{(break triangle along } \overline{PR}\text{)} \\
&= \frac{1}{2}\overline{QT} \cdot \overline{RP} + \frac{1}{2}\overline{RT'} \cdot \overline{RP} && \text{(area of triangles)} \\
&= \overline{QT} \cdot \overline{RP} && (\overline{QT} = \overline{RT'}) \\
&= \frac{b-a}{2}\left(\frac{b^2+a^2}{2} - \left(\frac{b+a}{2}\right)^2\right) && \text{(coordinates)} \\
&= \frac{b-a}{2}\left(\frac{b^2+a^2}{2} - \frac{b^2+2ba+a^2}{4}\right) && \text{(expansion)} \\
&= \frac{b-a}{2}\left(\frac{b^2-2ba+a^2}{4}\right) && \text{(cancel)} \\
&= \frac{b-a}{2}\left(\frac{b-a}{2}\right)^2 && \text{(factor)} \\
&= \boxed{\frac{1}{8}(b-a)^3} && \text{(combine)}
\end{aligned}
$$

$$
\begin{aligned}
A_{parseg} &= \int_a^b h(x)dx && \text{(area between line and parab} \\
&= \int_a^b (a+b)x + (a+b)a + a^2 - x^2 dx && \text{(def. } h \\
&= (a+b)\frac{x}{2} - (a+b)ax + a^2x - \frac{x^3}{3}\bigg|_a^b && \text{(calcu} \\
&= \frac{1}{2}(a+b)(b^2-a^2) - (a+b)a(b-a) - a^2(b-a) - \frac{1}{3}(b^3-a^3) && \text{(evaluat} \\
&= (b-a)\left[\frac{1}{2}(a+b)^2 - (a+b)a + a^2 - \frac{1}{3}(b^2+ba+a^2)\right] && \text{(common factor (} b - \\
&= (b-a)\left[\frac{1}{2}a^2 + ba + \frac{1}{2} + \frac{1}{2}b^2 - a^2 - ba + a^2 - \frac{1}{3}a^2 - \frac{1}{3}b^2\right] && \text{(expans} \\
&= (b-a)\left[\frac{1}{6}a^2 - \frac{1}{3}ba - \frac{1}{6}b^2\right] && \text{(can} \\
&= \boxed{\frac{1}{6}(b-a)^3} && \text{(fa}
\end{aligned}
$$

Thus, $A_{parseg} = \frac{4}{3}A_{\triangle PQQ'}$. □

5 Archimedes' Spiral

a.) Without using induction, show:

$$\sum_{\mathbf{k=1}}^{\mathbf{n}} \mathbf{k^2} = \frac{\mathbf{n^3}}{\mathbf{3}} + \frac{\mathbf{n^2}}{\mathbf{2}} + \frac{\mathbf{n}}{\mathbf{6}}$$

Consider the following telescoping sums of sequences.

$$n^3 = \sum_{k=1}^{n} k^3 - (k-1)^3$$
$$n^2 = \sum_{k=1}^{n} k^2 - (k-1)^2$$
$$n^1 = \sum_{k=1}^{n} k^1 - (k-1)^1$$

Now consider the expansion of the simple expression into sums over a common range.

$$\begin{aligned}
\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} &= \frac{1}{3}\left(\sum_{k=1}^{n} k^3 - (k-1)^3\right) + \frac{1}{2}\left(\sum_{k=1}^{n} k^2 - (k-1)^2\right) + \frac{1}{6}\left(\sum_{k=1}^{n} k - (k-1)\right) \\
&= \sum_{k=1}^{n}\left[\frac{1}{3}\left(k^3 - (k-1)^3\right) + \frac{1}{2}\left(k^2 - (k-1)^2\right) + \frac{1}{6}\left(k - (k-1)\right)\right] \\
&= \sum_{k=1}^{n}\left[\frac{1}{3}\left(3k^2 - 3k - 1\right) + \frac{1}{2}(2k+1) + \frac{1}{6}(1)\right] \\
&= \sum_{k=1}^{n}\left[k^2 - k - \frac{1}{3} + k + \frac{1}{2} + \frac{1}{6}\right] \\
&= \boxed{\sum_{k=1}^{n} k^2}
\end{aligned}$$

□

b.) Let Γ be the area inside of the spiral and Δ be the area of inside the whole disk shown in Figure 7. Using double contradiction, show:

$$\frac{\mathbf{\Gamma}}{\mathbf{\Delta}} = \frac{\mathbf{1}}{\mathbf{3}}$$

Begin by breaking up the spiral into n radial slices. In the arbitrary case, consider the area of the inscribed and circumscribed sectors. We know similar sectors are to

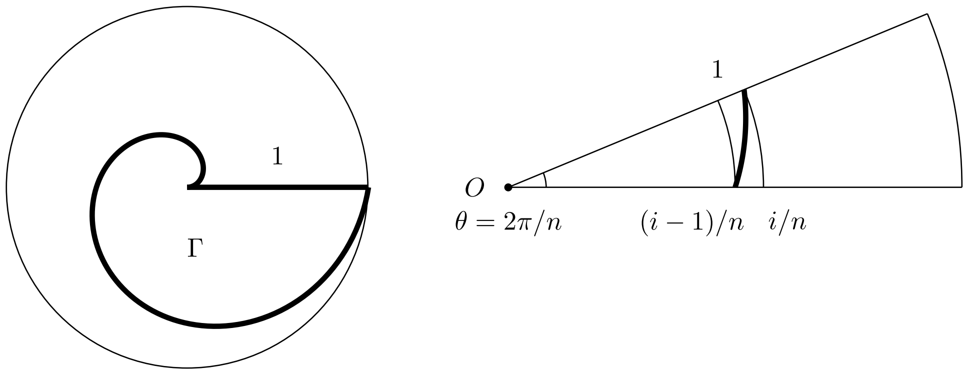

Figure 7: Archimedes' spiral with an arbitrary sector

one another as squares on their radii. Let δ be the area of a single, unit-sided, sector. Clearly, $n\delta = \Delta$ is the whole area of the circle. The sector with radius $\frac{i}{n}$ (the ith outer sector) has area $\delta\frac{i^2}{n^2}$ and the inner ($i-1$th) sector has area $\delta\frac{(i-1)^2}{n^2}$. Thus, the following expression represents the area of all of the circumscrubed sectors:

$$\begin{aligned}\Gamma &< \Gamma_{\text{above}}\\ &= \sum_{i=1}^{n} \delta\frac{i^2}{n^2}\\ &= \frac{\delta}{n^2}\sum_{i=1}^{n} i^2\\ &= \frac{\Delta}{n^3}\sum_{i=1}^{n} i^2\end{aligned}$$

And the inscribed sectors:

$$\begin{aligned}\Gamma &> \Gamma_{\text{below}} \\ &= \sum_{i=1}^{n} \delta \frac{(i-1)^2}{n^2} \\ &= \sum_{i=0}^{n-1} \delta \frac{i^2}{n^2} \\ &= \sum_{i=1}^{n-1} \delta \frac{i^2}{n^2} \\ &= \frac{\delta}{n^2} \sum_{i=1}^{n-1} i^2 \\ &= \frac{\Delta}{n^3} \sum_{i=1}^{n-1} i^2\end{aligned}$$

Or:

$$\frac{1}{n^3} \sum_{i=1}^{n-1} i^2 > \frac{\Gamma}{\Delta} > \frac{1}{n^3} \sum_{i=1}^{n} i^2$$

Now consider the result derived in the previous problem:

$$\begin{aligned}\sum_{i=1}^{n} i^2 &= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \\ \frac{1}{n^3} \sum_{i=1}^{n} i^2 &= \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} && > \frac{1}{3}\end{aligned}$$

And:

$$\begin{aligned}\sum_{i=1}^{n-1} i^2 &= \frac{(n-1)^3}{3} + \frac{(n-1)^2}{2} + \frac{n-1}{6} \\ \frac{1}{n^3} \sum_{i=1}^{n-1} i^2 &= \frac{1}{3} \frac{(n-1)^3}{n^3} + \frac{1}{2} \frac{(n-1)^2}{n^3} + \frac{1}{6} \frac{n-1}{n^3} \\ &< \frac{1}{3}\end{aligned}$$

Or:

$$\frac{1}{n^3} \sum_{i=1}^{n-1} i^2 < \frac{1}{3} < \frac{1}{n^3} \sum_{i=1}^{n} i^2$$

Thus, we find $\frac{\Gamma}{\Delta}$ and $\frac{1}{3}$ trapped between the same bounds. Futhermore, as n increases, we can become arbitrarily close to their targets. Let $L_n = \frac{1}{n^3} \sum_{i=1}^{n-1} i^2$ and $H_n =$

$\frac{1}{n^3}\sum_{i=1}^{n} i^2$. Suppose $\frac{\Gamma}{\Delta} < \frac{1}{3}$. We can find an n such that $\frac{\Gamma}{\Delta} < L_n < \frac{1}{3}$. However, as we previously proved, $L_n < \frac{\Gamma}{\Delta}$ but our implication claims the opposite, a contradiction. Now, suppose $\frac{\Gamma}{\Delta} > \frac{1}{3}$. Similarly, we can find an n such that $\frac{\Gamma}{\Delta} > H_n > \frac{1}{3}$. But $\frac{1}{3} < H_n$, contradiction! Thus, $\boxed{\frac{\Gamma}{\Delta} = \frac{1}{3}}$. $\square$

c.) Using upper and lower sums, show:

$$\int_{\mathbf{0}}^{\mathbf{B}} \mathbf{x^2 dx} = \frac{\mathbf{B^3}}{\mathbf{3}}$$

To begin, break interval from 0 to B into n equal length intervals with width $\frac{B}{n}$. Over each interval we can construct a rectangle that fits just under the parabola and one that fits just above. The total area of the rectangles above the parabola is just:

$$\begin{aligned} A_{\text{upper}} &= \frac{B}{n}\sum_{i=1}^{n}(B\frac{i}{n})^2 \\ &= \frac{B^3}{n^3}\sum_{i=1}^{n} i^2 \\ &> A \end{aligned}$$

And the area below is similarly:

$$\begin{aligned} A_{\text{lower}} &= \frac{B}{n}\sum_{i=1}^{n}(B\frac{i-1}{n})^2 \\ &= \frac{B^3}{n^3}\sum_{i=1}^{n}(i-1)^2 \\ &= \frac{B^3}{n^3}\sum_{i=1}^{n-1} i^2 \\ &< A \end{aligned}$$

Or:

$$\frac{1}{n^3}\sum_{i=1}^{n-1} i^2 < \frac{A}{B^3} < \frac{1}{n^3}\sum_{i=1}^{n} i^2$$

Recall from our previous result that:

$$\frac{1}{n^3}\sum_{i=1}^{n-1} i^2 < \frac{1}{3} < \frac{1}{n^3}\sum_{i=1}^{n} i^2$$

Observe that we are in the same situation, two bounds, abitrarily close, trap two expressions. So $\frac{1}{3}$ must equal $\frac{A}{B^3}$. Thus, after cross multiplication, it is clear $\boxed{A = \frac{B^3}{3}}$.
□

6 Rational Approximation to Square Roots

Set $\sqrt{c} = \sqrt{a^2 \pm b} = a \pm \delta$ where a is the natural number closest to c when squared.

a.) Show that $\sqrt{a^2 \pm b} < a \pm \frac{b}{2a}$.

Observe that $\delta^2 > 0$.

$$\begin{aligned}
a \pm \delta &= \sqrt{a^2 \pm b} && \text{(def.)} \\
(a \pm \delta)^2 &= a^2 \pm b && \text{(squaring)} \\
2a^2 \pm 2a\delta + \delta^2 &= 2a^2 \pm b && \text{(adding } a^2\text{)} \\
a \pm \delta + \frac{\delta^2}{2a} &= a \pm \frac{b}{2a} && \text{(dividing by } 2a\text{)} \\
a \pm \delta &< a \pm \frac{b}{2a} && \text{(replacing } \delta^2 \text{ by } 0\text{)} \\
\sqrt{a^2 \pm b} &< a \pm \frac{b}{2a} && \text{(def.)}
\end{aligned}$$

□

b.) Show that $a \pm \frac{b}{2a \pm 1} < \sqrt{a^2 \pm b}$.

Observe that $|\delta| < 1$, thus $\delta^2 < |\delta|$. When δ is negative $-\delta^2 > \delta$ or $\delta^2 < -\delta$. Let us express this relationship with the statement $\delta^2 < \pm\delta$.

$$\begin{aligned}
\sqrt{a^2 \pm b} &= a \pm \delta && \text{(def.)} \\
a^2 \pm b &= a^2 \pm 2a\delta + \delta^2 && \text{(squaring)} \\
\pm b &= \pm 2a\delta + \delta^2 && \text{(subtracting } a^2\text{)} \\
\pm b &< \pm 2a\delta \pm \delta && \text{(replacing } \delta^2 \text{ by } \pm\delta\text{)} \\
\pm b &< \pm(2a \pm 1)\delta && \text{(collecting terms)} \\
\pm \frac{b}{2a \pm 1} &< \pm\delta && \text{(dividing by } 2a \pm 1\text{)} \\
a \pm \frac{b}{2a \pm 1} &< a \pm \delta && \text{(adding } a\text{)} \\
a \pm \frac{b}{2a \pm 1} &< \sqrt{a^2 \pm b} && \text{(def.)}
\end{aligned}$$

□

c.) Apply this technique to confirm Archimedes' result for rational bounds on the value of π.

$$\mathbf{\frac{265}{153} < \sqrt{3} < \frac{1351}{780}}$$

Applying this technique directly to $\sqrt{3}$, we can find the bounds $\frac{5}{3} < \sqrt{3} < \frac{7}{4}$. These, however, do not match the Archimedian result. Let us now consider finding rational approximations to an integer multiple of $\sqrt{3}$. If we had rational bounds on $k\sqrt{3} = \sqrt{3k^2}$ we could turn these into bounds for $\sqrt{3}$ by simply dividing by k. Through computer search we have determined a promising value for k, namely $k = 15$. To confirm that this yields the Archimedian result, let us apply the rational approximation technique to $15\sqrt{3} = \sqrt{675}$

Let $a = 26$, $b = -1$. Surely $a^2 + b = 26 * 26 - 1 = 675$.

$$\begin{aligned}
a - \frac{b}{2a-1} &< \sqrt{675} < & a - \frac{b}{2a} \\
(26) - \frac{(1)}{2(26)-1} &< \sqrt{675} < & (26) - \frac{(1)}{2(26)} \\
26 - \frac{1}{51} &< \sqrt{675} < & 26 - \frac{1}{52} \\
\frac{1325}{51} &< \sqrt{675} < & \frac{1351}{52} \\
\frac{1325}{765} &< \sqrt{3} < & \frac{1351}{780} \\
&\text{finally}& \\
\frac{265}{152} &< \sqrt{3} < & \frac{1351}{780}
\end{aligned}$$

□

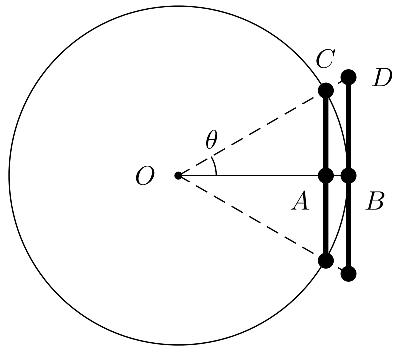

Figure 8: Sides of inscribed and circumscribed 6-gons

7 The Archimedian Approximation of π

Consider the following definitions relating to Figure 8:

$$p_n = \frac{\text{perimeter of inscribed } n\text{-gon}}{\text{diameter}}$$
$$q_n = \frac{\text{perimeter of circumscribed } n\text{-gon}}{\text{diameter}}$$

a.) Show that $p_n = n \sin \frac{\pi}{n}$ and $q_n = n \tan \frac{\pi}{n}$.

Observe that △COA and △DOB are similar, right triangles with opposite edges making up the sides of the inscribed and circumscribed n-gons respectively.

$$\begin{aligned} p_n &= \frac{\text{perimeter of inscribed } n\text{-gon}}{\text{diameter}} \\ &= \frac{2\text{n}\overline{CA}}{2\overline{OC}} \\ &= n\frac{\overline{CA}}{\overline{OC}} \\ &= \boxed{n \sin \frac{\pi}{n}} \end{aligned}$$

$$
\begin{aligned}
q_n &= \frac{\text{perimeter of circumscribed } n\text{-gon}}{\text{diameter}} \\
&= \frac{2\text{n}\overline{DB}}{2\overline{OB}} \\
&= n\frac{\overline{DB}}{\overline{OB}} \\
&= \boxed{n \tan \frac{\pi}{n}}
\end{aligned}
$$

□

b.) Show that $p_{2n} = \sqrt{q_{2n} p_n}$ and $q_{2n} = \frac{2q_n p_n}{q_n + p_n}$.

In the following derivations, let $\alpha = \frac{\pi}{2n}$.

$$
\begin{aligned}
\sqrt{q_{2n} p_n} &= \sqrt{(2n \tan \alpha)(n \sin 2\alpha)} && \text{(def.)} \\
&= \sqrt{2n^2 \tan \alpha \sin 2\alpha} && \text{(grouping)} \\
&= \sqrt{2n^2 \left(\frac{\sin \alpha}{\cos \alpha}\right)(2 \sin \alpha \cos \alpha)} && \text{(double angle theorems)} \\
&= \sqrt{2^2 n^2 \sin^2 \alpha} && \text{(canceling)} \\
&= 2n \sin \alpha && \text{(square rooting)} \\
&= 2n \sin \frac{\pi}{2n} && \text{(def. of } \alpha\text{)} \\
&= \boxed{p_{2n}} && \text{(def.)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{2q_n p_n}{q_n + p_n} &= \frac{2(n\tan 2\alpha)(n\sin 2\alpha)}{(n\tan 2\alpha) + (n\sin 2\alpha)} && \text{(def.)} \\
&= 2n\frac{\tan 2\alpha \sin 2\alpha}{\tan 2\alpha + \sin 2\alpha} && \text{(common factor } 2n\text{)} \\
&= 2n\frac{\frac{\sin^2 2\alpha}{\cos 2\alpha}}{\frac{\sin 2\alpha}{\cos 2\alpha} + \frac{\cos 2\alpha \sin 2\alpha}{\cos 2\alpha}} && \text{(trig.)} \\
&= 2n\frac{\sin 2\alpha}{1 + \cos 2\alpha} && \text{(multiply through by } \cos 2\alpha\text{)} \\
&= 2n\frac{2\sin\alpha\cos\alpha}{1 + 2\cos^2\alpha - 1} && \text{(double angle theorems)} \\
&= 2n\frac{\sin\alpha}{\cos\alpha} && \text{(canceling)} \\
&= 2n\tan\alpha && \text{(trig.)} \\
&= 2n\tan\frac{\pi}{2} && \text{(def. of } \alpha\text{)} \\
&= \boxed{q_{2n}} && \text{(def.)}
\end{aligned}
$$

□

c.) Assume $p_6 = 3$ and $q_6 = \frac{6}{\sqrt{3}}$. Compute the next few iterates.

1. Approximation using a 12-gon:

$$
q_{12} = \frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}} + 3}
$$

$$
p_{12} = \sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}} + 3}3}
$$

2. Approximation using a 24-gon:

$$
q_{24} = \frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3} + \sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}
$$

$$
p_{24} = \sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3} + \sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}} + 3}3}}
$$

3. Approximation using a 48-gon:

$$q_{48}=\frac{2\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}+\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}$$

$$p_{48}=\sqrt{\frac{2\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}+\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}$$

4. Approximation using a 96-gon:

$$q_{96}=\frac{2\frac{2\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}+\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}\sqrt{\frac{2\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}+\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}}{\frac{2\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}+\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}+\sqrt{\frac{2\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}+\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}}$$

$$p_{96}=\sqrt{\frac{2\frac{2\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}+\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}\sqrt{\frac{2\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}+\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}}{\frac{2\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}+\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}+\sqrt{\frac{2\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}+\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}}\sqrt{\frac{2\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}+\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}\sqrt{\frac{2\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}+\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}\sqrt{\frac{2\frac{6}{\sqrt{3}}3}{\frac{6}{\sqrt{3}}+3}3}}}}$$

Though the expression above is certainly not in simplest form, it is easy to appreciate the complexity of the expression when we observe that p_{96} involves the square root of an expression that involves the square root of an expression that involves the square root of an expression which involves the square root and expression that involves the square root of three.

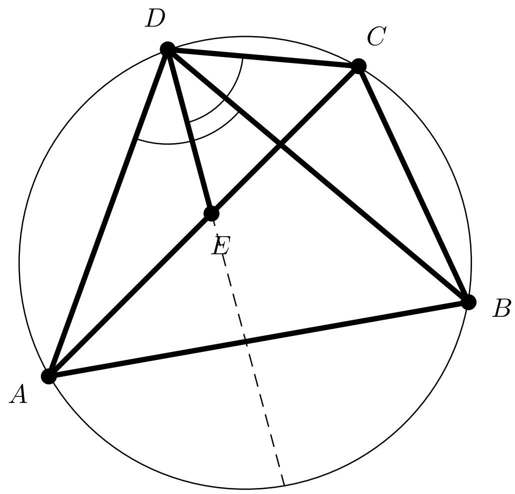

Figure 9: Ptolemy's Theorem.

8 Ptolemy's Inscribed Figures

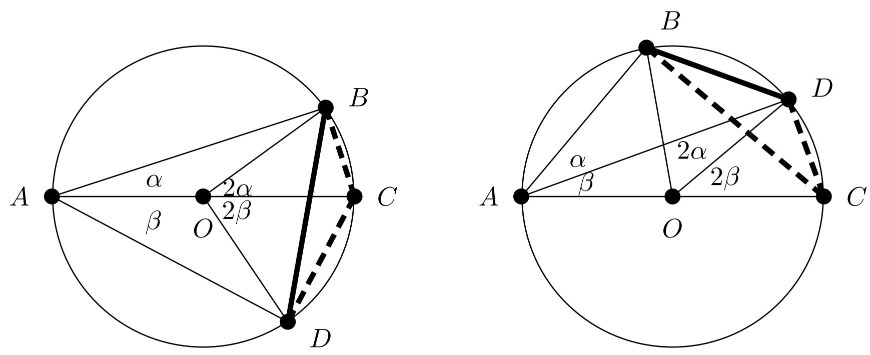

Figure 10: Sums and differences of angles.

9 Viete's Method for π

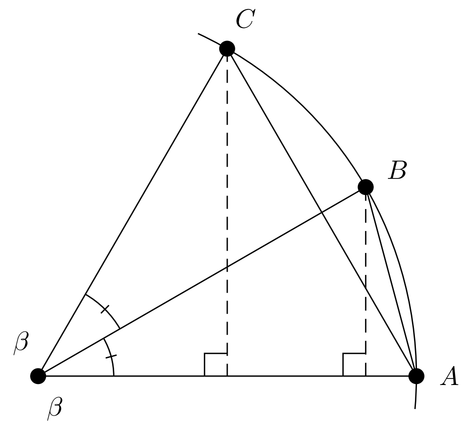

Figure 11: A slice of inscribed n and $2n$-gons.

10 Higher Parabolas

Suppose $y^q = kx^p$ where $p, q \geq 0$, so $y = k^{\frac{1}{q}} x^{\frac{p}{q}}$. Let $\gamma < 1$ be very close to one and $x_i = \gamma^i x_0$. These make up an infinite partition of the x-axis between 0 and B.

Observe that $y_i = k^{\frac{1}{q}} x_i^{\frac{p}{q}} = k^{\frac{1}{q}} \gamma^i x_0^{\frac{p}{q}}$.

a.) Compute A_1, A_2, and A_3 and find the general pattern.

$$A_1 = (x_0 - \gamma x_0)y_0 = (1-\gamma)x_0 y_0$$
$$A_2 = (\gamma x_0 - \gamma^2 x_0)y_1 = \gamma x_0(1-\gamma)y_0\gamma^{\frac{p}{q}} = \gamma^{\frac{p}{q}+1}(1-\gamma)x_0 y_0 = \gamma^{\frac{p}{q}+1}A_1$$
$$A_3 = (\gamma^2 x_0 - \gamma^3 x_0)y_2 = \gamma^2 x_0(1-\gamma)y_0(\gamma^2)^{\frac{p}{q}} = \gamma^{2(1+\frac{p}{q})}(1-\gamma)x_0 y_0 = \gamma^{2(1+\frac{p}{q})}A_1$$

So, the general pattern is the following.

$$\boxed{A_k = \gamma^{\left(1+\frac{p}{q}\right)(k-1)} A_1}$$

b.) Substitute $\beta = \gamma^{1+\frac{p}{q}}$ ($\beta < 1$ since $\gamma < 1$) and sum all of the rectangles.

Observe that succesive areas form a geometric sequence with ratio less than unity.

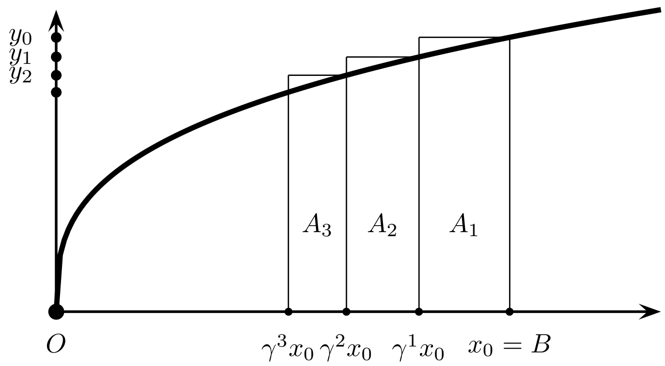

Figure 12: A higher parabola.

$$\begin{aligned}\mathcal{A} &= \sum_{k=1}^{\infty} A_k \\ &= \sum_{k=1}^{\infty} \beta^k A_1 \\ &= \frac{A_1}{1-\beta} \\ &= \frac{(1-\gamma)x_0 y_0}{1-\beta} \\ &= \boxed{\frac{1-\gamma}{1-\beta} k^{\frac{1}{q}} x_0^{1+\frac{p}{q}}}\end{aligned}$$

c.) Let $\gamma = \theta^q$. Find $\mathcal{A}$, not in terms of β.

Observe that $\beta = \gamma^{1+\frac{p}{q}} = (\theta^q)^{1+\frac{p}{q}} = \theta^{p+q}$. Using the fact $\frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + \cdots + x^n$, reconsider our expression for $\mathcal{A}$. Recall that $x_0 = B$.

$$
\begin{aligned}
\mathcal{A} &= \frac{1-\theta^{q}}{1-\theta^{p+q}} k^{\frac{1}{q}} B^{1+\frac{p}{q}} \\
&= \frac{1+\theta+\theta^{2}+\cdots+\theta^{q-1}}{1+\theta+\theta^{2}+\cdots+\theta^{p+q-1}} k^{\frac{1}{q}} x_{0}^{1+\frac{p}{q}} \\
(\text{as } \gamma \to 1) &= \frac{q}{p+q} k^{\frac{1}{q}} B^{1+\frac{p}{q}} \\
&= \boxed{\frac{1}{\frac{p}{q}+1} k^{\frac{1}{q}} B^{1+\frac{p}{q}} = \int_{0}^{B} k^{\frac{1}{q}} x^{\frac{p}{q}} dx}
\end{aligned}
$$

11 Napierian Logarithms

a.) Imagine two points racing along parallel line segments. The first point starts at $x(0) = 10^7$ and moves with velocity $x'(t) = -x(t)$. The second point starts at $y(0) = 0$ and moves with constant velocity $y(t) = 10^7$. At any point t, $y(t) = \mathbf{NL}[x(t)]$. Show that $y = -10^7 \ln[\frac{x}{10^7}]$.

Observe that $x(t)$ takes the form of a first-order, homogeneous, linear differential equation. Thus, we can immediately write down the form of its solution as $x(t) = Ce^{-t}$. Given the initial condition, we can conclude $\boxed{x(t) = 10^7 e^{-t}}$.

The function $y(t)$ has a constant differential, so clearly its solution takes the form $10^7 t + C$. Given the initial conditions, we can conclude $\boxed{y(t) = 10^7 t}$.

Now, writing y in terms of x using $t = \frac{y}{10^7}$ and simplifying notation:

$$
\begin{aligned}
x &= 10^7 e^{-t} \\
x &= 10^7 e^{-\frac{y}{10^7}} \\
\frac{x}{10^7} &= e^{-\frac{y}{10^7}} \\
\ln[\frac{x}{10^7}] &= -\frac{y}{10^7} \\
y &= -10^7 \ln[\frac{x}{10^7}] \qquad = \boxed{\mathrm{NL}[x]}
\end{aligned}
$$

b.) Show that $\mathbf{NL}[a] + \mathbf{NL}[b] \neq \mathbf{NL}[ab]$.

$$
\begin{aligned}
\mathrm{NL}[a] + \mathrm{NL}[b] &= \left(-10^7 \ln[\frac{a}{10^7}]\right) + \left(-10^7 \ln[\frac{b}{10^7}]\right) \\
&= -10^7 \left(\ln[\frac{a}{10^7}] + \ln[\frac{b}{10^7}]\right) \\
&= -10^7 \ln[\frac{ab}{10^{14}}] \\
&= \mathrm{NL}[a\frac{b}{10^7}] \neq \mathrm{NL}[ab]
\end{aligned}
$$

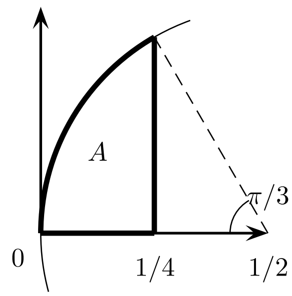

Figure 13: A portion of a circle

12 Finding π using the Binomial Expansion

a.) Show the area of A in Figure 13 is $\frac{\pi}{24} - \frac{\sqrt{3}}{32}$.

By geometry, we know the area of the entire circle in the figure is $\frac{\pi}{4}$. The sector containing A accounts for $\frac{1}{6}$th of the total area. The triangle adjacent to A, with base $\frac{1}{4}$ and height $\frac{\sqrt{3}}{4}$ has area $\frac{\sqrt{3}}{4}$. Thus, the area of A, the sector minus the triangle is clearly $\boxed{\frac{\pi}{24} - \frac{\sqrt{3}}{32}}$.

b.) Write an expression for the area of A as an integral and expand the integrand to four terms using the binomial expansion. Show $A = \frac{1}{12} - \frac{1}{160} - \frac{1}{3584} - \frac{1}{36864} - \cdots$.

We can find the area of A using calculus.

$$\begin{aligned} A &= \int_0^{\frac{1}{4}} \sqrt{x - x^2}dx \\ &= \int_0^{\frac{1}{4}} x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}dx \end{aligned}$$

Now, let us look at the expansion of $(1-x)^{\frac{1}{2}}$ by itself.

$$(1-x)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \prod_{i=0}^{k-1} \left(\frac{1}{2} - i\right)$$

Or, more concretely:

$$(1-x)^{\frac{1}{2}} \approx 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16}$$

Now, replacing this approximation into the expression for the area of A we get the following.

$$\begin{aligned}
A &\approx \int_0^{\frac{1}{4}} x^{\frac{1}{2}}\left(1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}\right)dx \\
&= \int_0^{\frac{1}{4}} x^{\frac{1}{2}}-\frac{x^{\frac{3}{2}}}{2}-\frac{x^{\frac{5}{2}}}{8}-\frac{x^{\frac{7}{2}}}{16}dx \\
&= \left[\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{5}x^{\frac{5}{2}}-\frac{1}{28}x^{\frac{7}{2}}-\frac{1}{72}x^{\frac{9}{2}}\right]_0^{\frac{1}{4}} \\
&= \boxed{\frac{1}{12}-\frac{1}{160}-\frac{1}{3584}-\frac{1}{36864}}
\end{aligned}$$

c.) Observe that $\sqrt{3}=\sqrt{\frac{4\cdot 3}{4}}=2\sqrt{\frac{3}{4}}=2\sqrt{1-\frac{1}{4}}$. Expand $\sqrt{1-\frac{1}{4}}$ to show $\sqrt{3}=2-\frac{1}{4}-\frac{1}{64}-\frac{1}{512}+\frac{5}{16384}$.

Consider the binomial expansion of $\sqrt{1-\frac{1}{4}}$.

$$\begin{aligned}
\left(1-\frac{1}{4}\right)^{\frac{1}{2}} &= \sum_{k=0}^{\infty}\frac{\left(-\frac{1}{4}\right)^k}{k!}\prod_{i=0}^{k-1}\left(\frac{1}{2}-i\right) \\
&= 1-\frac{1}{8}-\frac{1}{128}-\frac{1}{1024}-\frac{5}{32768}
\end{aligned}$$

So, $2\left(1-\frac{1}{4}\right)^{\frac{1}{2}}=\boxed{\sqrt{3}=2-\frac{1}{4}-\frac{1}{64}-\frac{1}{512}+\frac{5}{16384}}$.

d.) Combine this expression with the first result to solve for π.

We can take the geometric expression above and solve it for π.

$$\begin{aligned}
\pi &= 24\left(A+\frac{\sqrt{3}}{32}\right) \\
&= 24\left[\left(\frac{1}{12}-\frac{1}{160}-\frac{1}{3584}-\frac{1}{36864}\right)+\frac{\left(2-\frac{1}{4}-\frac{1}{64}-\frac{1}{512}+\frac{5}{16384}\right)}{32}\right] \\
&\approx \boxed{3.14219782}
\end{aligned}$$

13 Newton's Generalized Binomial Expansion

a.) Find Newton's expansion of $\log[1+x]$ using the binomial expansion.

$$\begin{aligned}
\log[1+x] &= \int_0^x \frac{1}{1+t}dt \\
&= \int_0^x (1+t)^{-1}dt \\
&= \int_0^z (1-t+t^2-t^3+t^4-t^5+\cdots)dt \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots \\
&= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{(k+1)}}
\end{aligned}$$

b.) Invert the previous result to find an expression for e^y.

Assume $x = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4$. Now, substituting:

$$\begin{aligned}
\log[1+x] = y &= 1 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots \\
&= 1 + (a_0 + a_1 y + a_2 y^2 + \cdots) - \frac{(a_0 + a_1 y + a_2 y^2 + \cdots)^2}{2} + \cdots
\end{aligned}$$

Observe that the left and right and hand sides of the equation above are polynomials in y. Because there is only a linear term on the left side, all of the *other* terms on the right hand side must cancel out. Now, using this fact, let us find all the ways we can make a constant term in the expansion of $\log[1+x]$ above:

$$\begin{aligned}
0 &= a_0 - \frac{a_0^2}{2} + \frac{a_0^3}{3} - \frac{a_0^4}{4} + \cdots \\
&= log[1+a_0]
\end{aligned}$$

And we know $\log[1] = 0$, so clearly, $\boxed{a_0 = 0}$.
Now, looking for the coefficients of linear terms, of which they must equal just 1 to match the single y on the left:

$$\begin{aligned}
1 &= a_1 - \frac{2a_0a_1}{2} + \frac{3a_0^2a_1}{3} - \frac{4a_0^3a_1}{4} + \cdots \\
&= a_1 - 0 + 0 - 0 + \cdots
\end{aligned}$$

So now we know $\boxed{a_1 = 1 = \frac{1}{1!}}$.
Similarly, we can collect the coefficients of quadratic terms, of which all must cancel out (recall that we found that there are really no constant terms, $a_0 = 0$):

$$\begin{aligned} 0 &= a_2 - \frac{a_1^2}{2} \\ &= a_2 - \frac{1}{2} \end{aligned}$$

This implies $\boxed{a_2 = \frac{1}{2} = \frac{1}{2!}}$.
Now, looking for the coefficients of cubic terms:

$$\begin{aligned} 0 &= a_3 - \frac{2a_1a_2}{2} + \frac{a_1^3}{3} \\ &= a_3 - \frac{1}{2} + \frac{1}{3} \\ &= a_3 - \frac{1}{6} \end{aligned}$$

So $\boxed{a_3 = \frac{1}{6} = \frac{1}{3!}}$.
Next, the quartic terms:

$$\begin{aligned} 0 &= a_4 - \frac{a_2^2 + 2a_1a_3}{2} + \frac{3a_1a_1a_2}{3} - \frac{a_1^4}{4} \\ &= a_4 - \frac{\frac{1}{4} + \frac{1}{3}}{2} + \frac{1}{2} - \frac{1}{4} \\ &= a_4 - \frac{7}{24} + \frac{12}{24} - \frac{6}{24} \\ &= a_4 - \frac{1}{24} \end{aligned}$$

So $\boxed{a_4 = \frac{1}{24} = \frac{1}{4!}}$. Continuing in this pattern we can conjecture the following expansion for x.

$$x = \sum_{k=1}^{\infty} \frac{y^k}{k!}$$

Putting this result back into the relation $\log[1+x] = y$ we can conclude the follow expansion of the exponential function (note that the sum starts at $k=0$ now).

$$\boxed{e^y = \sum_{k=0}^{\infty} \frac{y^k}{k!}}$$

c.) Find the binomial expansion of $[1+\frac{1}{n}]^n$. Then let $n \to \infty$ to find an expression for e.

Directly applying the binomial expansion to yields the following.

$$\begin{aligned}
\left[1+\frac{1}{n}\right]^n &= 1\left(\frac{1}{n}\right)^0 + n\left(\frac{1}{n}\right)^1 + \frac{n(n-1)}{2}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \cdots + 1\left(\frac{1}{n}\right)^n \\
&= 1 + 1 + \frac{1-\frac{1}{n}}{2} + \frac{1-\frac{3}{n}+\frac{2}{n^2}}{3!} + \cdots + 1\frac{1}{n^n} \\
&= \boxed{1 + 1 + \frac{1-\frac{1}{n}}{2} + \frac{(1-\frac{1}{n})(1-\frac{2}{n})}{3!} + \cdots + \frac{(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{n-1}{n})}{n!}}
\end{aligned}$$

If we plug in ∞ for n in the above all fractions involving n go to zero and can be dropped. Thus, we are left with this expression for e.

$$\lim_{n\to\infty}\left[1+\frac{1}{n}\right]^n = \sum_{k=0}^{\infty}\frac{1}{k!} = e$$

d.) Next, show the previous result is identical to e^x.

Simply replacing 1 by x in the previous derivation we find the following relation.

$$\lim_{n\to\infty}\left[1+\frac{x}{n}\right]^n = \sum_{k=0}^{\infty}\frac{x^k}{k!}$$

But this just the same formula for e^y we found before using an x instead!

e.) Finally, show that $\log[x]$ and e^x are inverses.

If $\log[x]$ and e^x are inverses, $\log[e^x]$ should be simply x. Let us apply our previously found expansions.

$$
\begin{aligned}
\log[e^x] &= \log\left[\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\right] \\
&= \lim_{n\to\infty}\log\left[\left(1+\frac{x}{n}\right)^n\right] \\
&= \lim_{n\to\infty} n\log\left[1+\frac{x}{n}\right] \\
&= \lim_{n\to\infty} n\sum_{k=0}^{\infty}\frac{(-1)^k \frac{x}{n}^{k+1}}{(k+1)!} \\
&= \lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{(-1)^k \frac{x^{k+1}}{n^k}}{(k+1)!} \\
&= x+\lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{(-1)^k \frac{x^{k+1}}{n^k}}{(k+1)!} \\
&= \boxed{x}
\end{aligned}
$$

14 The magic number π

a.) Set $x = \frac{\pi}{2}$ in the equation $\frac{\sin x}{x} = \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2\pi^2}\right)$.

$$\sin\frac{\pi}{2}\frac{\pi}{2} = \frac{2}{\pi} = \prod_{k=1}^{\infty}\left(1-\frac{4}{k^2}\right)$$
$$= \prod_{k=1}^{\infty}\frac{(2k-1)(2k+1)}{(2k)^2}$$
$$= \frac{1}{2}\frac{3}{2}\frac{3}{4}\frac{5}{4}\frac{5}{6}\frac{7}{6}\cdots$$

The final result is exactly the Wallis Product!

b.) Now, replace $\frac{\sin x}{x}$ with $\cos x$.

Let $P(x) = \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$. The roots of $P(x)$ (by inspection), are $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$. Therefore, $P(x) = (1 - \frac{2x}{\pi})(1 + \frac{2x}{\pi})(1 - \frac{2x}{3\pi})(1 + \frac{2x}{3\pi})\cdots$. Or, after collecting pairs, we can see this formulation. $P(x) = (1 - frac4x^2\pi^2)(1 - \frac{4x^2}{9\pi^2})\cdots$. To simplify, let $z = \frac{4x^2}{\pi^2}$. Now, $P(z) = (1-z)(1-\frac{z}{3^2})(1-\frac{z}{5^2})\cdots$. Collecting only the terms linear in z after the product yields $\frac{1}{2}\frac{\pi^2}{4} = \boxed{\frac{pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}}$.

Interestingly, this is exactly the sum of the inverses of the odd squares!